Math 429 - Exercise Sheet 9

1. We already saw the root decomposition of $\mathfrak{sl}_n$ (it is called "type A"). Work out the root decomposition of $\mathfrak{g} \in \{\mathfrak{o}_{2n+1}, \mathfrak{sp}_{2n}, \mathfrak{o}_{2n}\}$ (which are called "types B,C,D", respectively) by starting from the Cartan subalgebra of $\mathfrak{g}$ consisting of diagonal matrices.

Solution. We begin by setting $\mathfrak{g} = \mathfrak{o}_{2n}$. We denote as $A = (a_1, \ldots, a_n)$ the 2×2 block-diagonal matrices whose kth block is

$$\begin{bmatrix} 0 & a_k \\ -a_k & 0 \end{bmatrix}, \tag{1}$$

for some $a_k \in \mathbb{C}$. Let $\mathfrak{h}$ be the toral subalgebra which consists of all matrices $A = (a_1, \ldots, a_n)$ as above. A basis for the dual $\mathfrak{h}^*$ is given by the linear maps $\alpha_k \colon A = (a_1, \ldots, a_n) \mapsto a_k$.
The common eignvectors for the adjoint action of $\mathfrak{h}$ are described as follows. Let

$$C^1 = \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix}, C^2 = \begin{bmatrix} 1 & -i \\ -i & -1 \end{bmatrix}, C^3 = \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}, C^4 = \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}.$$

For every pair $1 \leq r < j \leq n$ and $k = 1, 2, 3, 4$, let $C^k_{(r,j)}$ be the 2×2 block matrix having the matrix C^k in the (r, j)th block and $-(C^k)^T$ in the (j, r)th block. Then for every $A = (a_1, \ldots, a_n) \in \mathfrak{h}$ we have

$$[A, C^k_{(r,j)}] = C^k_{(r,j)} \cdot \begin{cases} i(a_r + a_j) & k = 1 \\ -i(a_r + a_j) & k = 2 \\ i(a_r - a_j) & k = 3 \\ -i(a_r - a_j) & k = 4 \end{cases}. \tag{2}$$

Then we have the decomposition in $\mathrm{ad}_{\mathfrak{h}}$-eigenspaces

$$\mathfrak{o}_{2n} = (\mathfrak{o}_{2n})_0 \bigoplus_{1 \leq r < j \leq n, k=1,2,3,4} \mathbb{C}C^k_{(r,j)}, \tag{3}$$

where $\mathbb{C}C^k_{(r,j)}$ is the eigenspace associated to the eigenvalue $\pm i(\alpha_r \pm \alpha_j)$ (with signs according to (2)), and $(\mathfrak{o}_{2n})_0$ is the centralizer of $\mathfrak{h}$ in $\mathfrak{g}$. Finally, for dimensional reasons in (3), we have

$$(\mathfrak{o}_{2n})_0 = \mathfrak{h},$$

so that $\mathfrak{h}$ is *maximal* as a toral subalgebra. Thus (3) is the root decomposition of $\mathfrak{o}_{2n}$.

Let $\mathfrak{g} = \mathfrak{o}_{2n+1}$. A Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{o}_{2n+1}$ is given by block diagonal matrices having n consecutive 2×2 blocks of the form (1), and a 1×1 block which we set to be 0. Then, every root for $\mathfrak{o}_{2n}$ is also a root for $\mathfrak{o}_{2n+1}$, and there are $2n$ more roots which we now describe. The first set of n eigenvectors is given by matrices having

$$B_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

in entries $(2k, 2n+1)$ and $(2k+1, 2n+1)$, and having $-B_1^T$ in entries $(2n+1, 2k+1)$ and $(2n+1, 2k)$. The second set of n eigenvectors is given by matrices having

$$B_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

in entries $(2k, 2n+1)$ and $(2k+1, 2n+1)$, and having $-B_1^T$ in entries $(2n+1, 2k+1)$ and $(2n+1, 2k)$. The corresponding eigenvalues are $i\alpha_k$ and $-i\alpha_k$, with the same notations as above. Once again we deduce from the eigenspace decomposition just described that $\mathfrak{t}$ is in fact a Cartan subalgebra.

Let $\mathfrak{g} = \mathfrak{sp}_{2n}$. Recall that elements $X \in \mathfrak{g}$ are of the form

$$X = \begin{bmatrix} A & B \\ C & -A^T \end{bmatrix}$$

where B and C are symmetric matrices. We take the Cartan subalgebra $\mathfrak{h}$ consisting of diagonal matrices in $\mathfrak{sp}_{2n}$. Let $\alpha_k \in \mathfrak{h}^*$ be the functional which takes the kth entry in the diagonal, for $k = 1, \dots, n$.
Let $E_{k,l}$ denote the matrix whose only nonzero entry is a 1 in position (k, l). Then for $1 \leq k < l \leq n$

$$\begin{bmatrix} 0 & E_{k,l} + E_{l,k} \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ E_{k,l} + E_{l,k} & 0 \end{bmatrix}$$

are eigenvectors for the eigenvalues $\alpha_k + \alpha_l$ and $\alpha_k + \alpha_l$ respectively. Matrices of the form

$$\begin{bmatrix} E_{k,l} & 0 \\ 0 & -E_{l,k} \end{bmatrix},$$

for $k < l$ are eigenvectors for the eigenvalues $\alpha_k - \alpha_l$. Finally,

$$\begin{bmatrix} 0 & E_{k,k} \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ E_{k,k} & 0 \end{bmatrix}$$

are eigenvectors for the eigenvalues $2\alpha_k$ and $-2\alpha_k$ respectively. As before we deduce from the eigenspace decomposition just described that $\mathfrak{h}$ is in fact a Cartan subalgebra.

2. Compute explicitly the $\mathfrak{sl}_2$-triples associated to the root decompositions of $\mathfrak{o}_{2n+1}, \mathfrak{sp}_{2n}, \mathfrak{o}_{2n}$ you found in the previous part.

Solution. We begin by setting $\mathfrak{g} = \mathfrak{o}_{2n}$, and we choose the trace form $\operatorname{tr}(XY)$ as a *s.i.b.f.*. With the same notations as above, we compute the $\mathfrak{sl}_2$ triple associated to the root $\alpha = i(\alpha_r + \alpha_j) \in \mathfrak{h}^*$ (the other ones being analogous). The matrix

$$h_\alpha = \frac{-i}{2}\left(0, \dots, \overbrace{1}^{r\text{th position}} \dots, \overbrace{1}^{j\text{th position}}, \dots, 0\right)$$

is such that

$$\operatorname{tr}(h_\alpha \cdot (a_1, \dots, a_n)) = i(a_j + a_r) = \alpha((a_1, \dots, a_n)).$$

for all $(a_1, \dots, a_n) \in \mathfrak{h}$. Moreover,

$$(\alpha, \alpha) = \operatorname{tr}(h_\alpha \cdot h_\alpha) = \alpha(h_\alpha) = 1.$$

Following Proposition 19 in the Lecture notes, we set $H_\alpha = 2h_\alpha$. The root space associated to the root α is $\mathbb{C}C^1_{(r,j)}$, and the one associated to the root $-\alpha$ is $\mathbb{C}C^2_{(r,j)}$. Then we choose $E_\alpha = C^1_{(r,j)}$ and we compute

$$(C^1_{(r,j)}, C^2_{(r,j)}) = \operatorname{tr}(C^1_{(r,j)} \cdot C^2_{(r,j)}) = 8.$$

Thus Proposition 19 of the Lecture notes tells us to set $F_\alpha = \frac{-i}{8} C^2_{(r,j)}$, and $(E_\alpha, H_\alpha, F_\alpha)$ is the $\mathfrak{sl}_2$ triple.

In the case $\mathfrak{g} = \mathfrak{o}_{2n+1}$, we take the same *s.i.b.f.* and we get analogous computations as for $\mathfrak{o}_{2n}$. The dual elements to the additional roots $\pm\alpha_k$ are

$$\tilde{h}_{\pm\alpha_k} = \pm\frac{i}{2}\left(0, \ldots, \overbrace{1}^{k\text{th position}}, \ldots, 0\right).$$

Let $\mathfrak{g} = \mathfrak{sp}_{2n}$. We compute a $\mathfrak{sl}_2$ triple associated to the root $\alpha = i(\alpha_k + \alpha_l)$ for $k \neq l$ by means of the *s.i.b.f.* $\operatorname{tr}(XY)$. The equality of linear operators on $\mathfrak{h}$

$$\operatorname{tr}\left(\frac{i}{2}\begin{bmatrix} E_{k,k} + E_{l,l} & 0 \\ 0 & -E_{k,k} - E_{l,l} \end{bmatrix} \cdot -\right) = i(\alpha_k + \alpha_l)(-)$$

and the equality of numbers

$$\operatorname{tr}\left(\frac{i}{2}\begin{bmatrix} E_{k,k} + E_{l,l} & 0 \\ 0 & -E_{k,k} - E_{l,l} \end{bmatrix} \cdot \frac{i}{2}\begin{bmatrix} E_{k,k} + E_{l,l} & 0 \\ 0 & -E_{k,k} - E_{l,l} \end{bmatrix}\right) = 2i$$

tell us to set

$$H_\alpha = \begin{bmatrix} E_{k,k} + E_{l,l} & 0 \\ 0 & -E_{k,k} - E_{l,l} \end{bmatrix}.$$

Next we take a nonzero vector in the root space for the root α

$$E_\alpha = \begin{bmatrix} 0 & E_{k,l} + E_{l,k} \\ 0 & 0 \end{bmatrix},$$

and we look for the normalization for a vector in the root space associated to $-\alpha$. Explicitely, we have

$$\operatorname{tr}\left(\begin{bmatrix} 0 & E_{k,l} + E_{l,k} \\ 0 & 0 \end{bmatrix} \cdot \frac{-i}{2}\begin{bmatrix} 0 & 0 \\ E_{k,l} + E_{l,k} & 0 \end{bmatrix}\right) = -i = \frac{2}{(\alpha, \alpha)},$$

and we set

$$F_\alpha = \frac{-i}{2}\begin{bmatrix} 0 & 0 \\ E_{k,l} + E_{l,k} & 0 \end{bmatrix}.$$

3. Show that any 3-dimensional complex semisimple Lie algebra is isomorphic to $\mathfrak{sl}_2$.

Solution. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, which exists as a maximal toral subalgebra. Since the center of the semisimple Lie algebra $\mathfrak{g}$ is trivial, one can find a non zero root α in the root decomposition with respect to $\mathfrak{h}$. By dimensional reasons, the $\mathfrak{sl}_2$ triple associated to α provides an isomorphism $\mathfrak{sl}_2 \cong \mathfrak{g}$.

4. Show that the root spaces of any complex semisimple Lie algebra $\mathfrak{g}$ satisfy

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$$

whenever the roots α, β satisfy $\alpha + \beta \neq 0$ (the inclusion $\subseteq$ is obvious, it's $\supseteq$ that's tricky).

Solution. We already know that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta]$ is contained in the one dimensional subspace $\mathfrak{g}_{\alpha+\beta}$. Thus we have to show that for all roots α, β such that $\alpha + \beta$ is a nonzero root, $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \neq 0$. Let $S_\alpha = \langle E_\alpha, H_\alpha, F_\alpha \rangle \cong \mathfrak{sl}_2$ be the $\mathfrak{sl}_2$ triple associated to the root α by Proposition 19 in the Lecture notes, and let $K = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_{\alpha + i\beta}$. Of course only finitely many summands will be non-zero. Observe that K is a S_α submodule of $\mathfrak{g}$. In order to prove that K is irreducible we use the following fact, which is easy to check.

Lemma 1. *Let V be a $\mathfrak{sl}_2$ representation. The number of irreducible summands in the decomposition $V \cong L(n_1) \oplus \cdots \oplus L(n_k)$ is*

$$k = \dim V_1 + \dim V_0$$

where $V_j = \{v \in V | Hv = jv\}$.

According to Proposition 20 in the Lecture notes, each summand $\mathfrak{g}_{\alpha+i\beta}$ is a one dimensional weight space associated to the integral weight

$$(i\alpha + \beta)(H_\alpha) = 2i + \beta(H_\alpha).$$

It follows that only one of 0 and 1 can occur as weight. Thus the above Lemma implies that K is irreducible. In particular, the adjoint action of $\mathfrak{g}_\alpha$ maps $\mathfrak{g}_\beta$ onto $\mathfrak{g}_{\alpha+\beta}$.

5. Using the result below as a black box, show that indeed the elements H_α defined in Lecture do not depend on the choice of s.i.b.f. on a complex semisimple Lie algebra $\mathfrak{g}$.

Solution. Assume that the Lie algebra $\mathfrak{g}$ is simple. We know that any pair of *s.i.b.f.*'s $(-,-)$ and $(-,-)'$ are equal up to a scalar multiple

$$(-,-) = \lambda(-,-)'. \tag{4}$$

Let H_α and H'_α be the elements defined in the Lecture notes w.r.t. the forms $(-,-)$ and $(-,-)'$ respectively. More precisely

$$H_\alpha = \frac{2}{\alpha(h_\alpha)} h_\alpha \text{ and } H'_\alpha = \frac{2}{\alpha(h'_\alpha)} h'_\alpha$$

where the elements h_α and h'_α are defined by $(h_\alpha, -) = \alpha(-) = (h'_\alpha, -)'$. It follows from (4) that $h'_\alpha = \lambda h_\alpha$, and then $H_\alpha = H'_\alpha$. Finally, if the Lie algebra $\mathfrak{g}$ is semisimple, Lemma 5 in Lecture 8 of the Lecture notes tells us that we can decompose it as a sum of simple Lie algebras, and the statement (*) below implies that we can perform the above proof on each simple summand separately.

(*) If $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ with $\mathfrak{g}_1, \ldots, \mathfrak{g}_k$ simple, show that any Cartan subalgebra in $\mathfrak{g}$ is the direct sum of a collection of Cartan subalgebras in $\mathfrak{g}_1, \ldots, \mathfrak{g}_k$. In this case, also conclude that the root decomposition of $\mathfrak{g}$ is the corresponding direct sum of the root decompositions of $\mathfrak{g}_1, \ldots, \mathfrak{g}_k$.

Hi all,

I am writing this post because I want to point out a mistake that I made while talking with some of you.

Let $\mathfrak{h}$ be a (non maximal) toral subalgebra of a Lie algebra $\mathfrak{g}$. By Proposition 16 in the Lecture notes, we have an eigenspaces decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \bigoplus_{\lambda \in \mathfrak{h}^*} \mathfrak{g}_\lambda, \tag{5}$$

but **the dimensions of the $\mathfrak{g}_\lambda$'s is not one in general**. This is a coarser decomposition, and here is an example by Andrei.

Let $\mathfrak{g} = \mathfrak{sl}_4$ and let $\mathfrak{h}$ be the one dimensional toral subalgebra consisting of matrices

$$\begin{bmatrix} x & 0 & & \\ 0 & x & & \\ & & -x & 0 \\ & & 0 & -x \end{bmatrix}. \tag{6}$$

The decomposition (5) is then

$$\mathfrak{sl}_4 = (\mathfrak{sl}_4)_0 \oplus (\mathfrak{sl}_4)_1 \oplus (\mathfrak{sl}_4)_{-1},$$

where the summands are described as follows.

- $(\mathfrak{sl}_4)_0$ is the 7 dimensional centralizer of $\mathfrak{h}$, given by block matrices of the form $\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$ in $\mathfrak{sl}_4$.
- $(\mathfrak{sl}_4)_1$ is the 4 dimensional eigenspace consisting of matrices of the form $\begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix}$. The associated eigenvalue sends the matrix (6) to $2x$.
- $(\mathfrak{sl}_4)_{-1}$ is the 4 dimensional eigenspace consisting of matrices of the form $\begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix}$. The associated eigenvalue sends the matrix (6) to $-2x$.

If you go through the proof of Proposition 20 in the lecture notes, you can see that the maximality of the toral subalgebra is actually needed. Indeed, the restriction of our s.i.b.f. to $\mathfrak{g}_0$ is going to be non degenerate, but the further restriction to the toral subalgebra will be degenerate. As a result, the dual h_α to a given root α (defined in paragraph 9.4) will not belong to the toral subalgebra in general, and Proposition 19 fails.

In any case, this remark does not change the solution of the first exercise in Sheet 9. Finding enough (one dimensional) root spaces allows one to conclude that $\dim \mathfrak{g}_0 = \dim \mathfrak{h}$ in (5), thus implying that the toral subalgebra $\mathfrak{h}$ is maximal.

Sorry for the possible confusion, I hope things are clear now. I wish you a happy Easter and a nice break,

Niccolò